EXISTENCE THEOREM FOR WEAK QUASIPERIODIC SOLUTIONS OF LAGRANGIAN SYSTEMS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We establish new sufficient conditions for the existence of weak Besicovitch quasiperiodic solutions for natural Lagrangian system on Riemannian manifold with time-quasiperiodic force function.

1. Introduction. Let $\mathcal{M}$ be a smooth complete connected m-dimensional Riemannian manifold equipped with an inner product $\langle\cdot,\cdot\rangle$ on fibers $T_x\mathcal{M}$ of tangent bundle $T\mathcal{M}$. Consider a natural system on $\mathcal{M}$ with Lagrangian function of the form $L\big|_{T_x\mathcal{M}}= \frac{1}{2}\langle\dot{x},\dot{x}\rangle - \Pi(t,x)$ where $\frac{1}{2}\langle\dot{x},\dot{x}\rangle$ and $\Pi(t,x)$ stand for kinetic and potential energy respectively. We suppose that the potential energy is represented as $\Pi := -W(\omega t,x)$ where $W(\omega t,x)$ is ω-quasiperiodic force function generated by a function $W(\cdot,\cdot)\in \mathrm{C}^{0,2}(\mathbb{T}^k\times\mathcal{M}\mapsto\mathbb{R})$ ($W(\cdot,\cdot)$ is continuous together with $W''_{xx}(\cdot,\cdot)$); here $\mathbb{T}^k=\mathbb{R}^k/2\pi\mathbb{Z}^k$ is k-dimensional torus and $\omega=(\omega_1,...,\omega_k)\in\mathbb{R}^k$ is a frequencies vector with rationally independent components. The problem is to detect in such a system ω-quasiperiodic oscillations.

J. Blot in his series of papers [1–4] applied variational method to establish the existence of weak almost periodic solutions for systems in $\mathbb{E}^m$. Later, this method was used in [5–8] to prove the existence of weak and classical almost periodic solutions for systems of variational type. In [9, 10], weak and classical quasiperiodic solutions were found for natural mechanical systems in convex compact subsets of Riemannian manifolds with non-positive sectional curvature. The goal of the present paper is to extend these results to natural systems on arbitrary Riemannian manifolds.

2. Variational method. One can interpret a natural system on $\mathcal{M}$ as a natural system in Euclidean space $\mathbb{E}^n$ (of appropriate dimension n) with holonomic constraint. Namely, in view of the Nash embedding theorem [11] we consider $\mathcal{M}$ as a submanifold of $\mathbb{E}^n$ for some natural $n>m$. The set $\mathcal{M}\subset\mathbb{E}^n$ play the role of holonomic constraint for natural system in $\mathbb{E}^n$ with kinetic energy $K=\frac{1}{2}\langle\dot{y},\dot{y}\rangle_{\mathbb{E}^n}$ and potential energy $-W(\omega t,y)$, if we suppose that $W(\cdot,\cdot)$ is defined in $\mathbb{T}^k\times\mathbb{E}^n$.

In what follows we shall use identical notations for inner product $\langle\cdot,\cdot\rangle_{\mathbb{E}^n}$ of $\mathbb{E}^n$ and the induced inner product $\langle\cdot,\cdot\rangle$ on $T\mathcal{M}$. Let ∇_ξ stands for the covariant differentiation of Levi-Civita connection in the direction of vector

$\xi \in T\mathcal{M}$, and let ∇f stands for gradient vector field of a scalar function $f(\cdot) : \mathcal{M} \mapsto \mathbb{R}$, i.e $\langle \nabla f(x), \xi \rangle = \mathrm{d}f(x)(\xi)$ for any $\xi \in T_x\mathcal{M}$.

Denote by $\mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ the space of $\mathbb{E}^n$-valued functions on k-torus which are integrable with the square of Euclidean norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. Define on $\mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ the standard scalar product $\langle \cdot, \cdot \rangle_0 = (2\pi)^{-k} \int_{\mathbb{T}^k} \langle \cdot, \cdot \rangle \mathrm{d}\varphi$ and the corresponding semi-norm $\|\cdot\|_0 := \sqrt{\langle \cdot, \cdot \rangle_0}$. By $\mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$ denote the space of functions $f(\cdot) \in \mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ each of which has weak (Sobolev) derivative $D_\omega f(\cdot) \in \mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ in the direction of vector ω. Recall that a function $u(\cdot) \in \mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ with Fourier series $\sum_{\mathbf{n} \in \mathbb{Z}^k} u_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\varphi}$ has a weak derivative iff the series $\sum_{\mathbf{n} \in \mathbb{Z}^k} |\mathbf{n} \cdot \omega|^2 \|u_{\mathbf{n}}\|^2$ converges and then the Fourier series of $D_\omega u(\cdot)$ is $\sum_{\mathbf{n} \in \mathbb{Z}^k} \mathrm{i}(\mathbf{n} \cdot \omega) u_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\varphi}$.

The space $\mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$ is equipped with the semi-norm $\|\cdot\|_1$ generated by the scalar product $\langle D_\omega \cdot, D_\omega \cdot \rangle_0 + \langle \cdot, \cdot \rangle_0$. After identification of functions coinciding a.e., both spaces becomes Hilbert spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively.

To any function $u(\cdot) \in \mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ with Fourier series $\sum_{\mathbf{n} \in \mathbb{Z}^k} u_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\varphi}$, one can put into correspondence a Besicovitch quasiperiodic function $x(t) = u(\omega t)$ defined by its Fourier series $\sum_{\mathbf{n} \in \mathbb{Z}^k} u_{\mathbf{n}} \mathrm{e}^{\mathrm{i}(\mathbf{n}\cdot\omega)t}$. If $u(\cdot) \in \mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$ then $\dot{x}(t)$ denotes a Besicovitch quasiperiodic function $D_\omega u(\omega t)$.

We define weak solution of Lagrangian system on $\mathcal{M}$ with density $L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle + W(\omega t, x)$ in a slightly different way then in [7]. First, for any bounded subset $\mathcal{A} \subseteq \mathcal{M}$, put

$$\mathcal{S}_{\mathcal{A}} := \mathrm{C}^\infty \left(\mathbb{T}^k \mapsto \mathcal{A} \right).$$

Observe that if $u_j(\cdot) \in \mathcal{S}_{\mathcal{A}}$ is a sequence bounded in $\mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$ and convergent to a function $u(\cdot)$ by norm of the space $\mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ (recall that we consider the set $\mathcal{A} \subseteq \mathcal{M}$ both as a subset of $\mathbb{E}^n$), then for any $\mathbf{n} \in \mathbb{Z}^k$ the sequence of Fourier series coefficients $u_{\mathbf{n},j}$ converges to $u_{\mathbf{n}}$ and for some $K > 0$ we have

$$\sum_{|\mathbf{n}| \le N} |\mathbf{n} \cdot \omega|^2 \|u_{\mathbf{n}}\|^2 = \lim_{j \to \infty} \sum_{|\mathbf{n}| \le N} |\mathbf{n} \cdot \omega|^2 \|u_{j,\mathbf{n}}\|^2 \le$$
$$\le \liminf_{j \to \infty} \sum_{\mathbf{n} \in \mathbb{Z}^k} |\mathbf{n} \cdot \omega|^2 \|u_{j,\mathbf{n}}\|^2 \le K \quad \forall N \in \mathbb{N}.$$

Hence, $u(\cdot) \in \mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$ and $\|D_\omega u\|_0 \le \liminf_{j \to \infty} \|D_\omega u_j\|_0$. Moreover, $u_j(\cdot)$ converges to $u(\cdot)$ weakly in $\mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$.

Next, for any bounded subset $\mathcal{A} \subseteq \mathcal{M}$ define a functional space $\mathcal{H}_{\mathcal{A}}$ in a following way: $u(\cdot) \in \mathcal{H}_{\mathcal{A}}$ iff there exists a sequence $u_j(\cdot) \in \mathcal{S}_{\mathcal{A}}$ bounded in $\mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$ and convergent to $u(\cdot)$ by norm of the space $\mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ (recall that we consider the set $\mathcal{A} \subseteq \mathcal{M}$ both as a subset of $\mathbb{E}^n$). As it was noted above $\mathcal{H}_{\mathcal{A}} \subset \mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$. We shall say that $h(\cdot) \in \mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$ is a vector field along the map $u(\cdot) \in \mathcal{H}_{\mathcal{A}}$ defined in the above sens by a sequence $u_j(\cdot)$ if there exists a sequence $h_j(\cdot) \in \mathrm{C}^\infty \left(\mathbb{T}^k \mapsto T\mathcal{M} \right)$ such that

$h_j(\varphi) \in T_{u_j(\varphi)}\mathcal{M}$, the sequences $\max_{\varphi\in\mathbb{T}^k} \|h_j(\varphi)\|$, $\|h_j\|_1$ are bounded, and $\lim_{j\to\infty} \|h - h_j\|_1 = 0$.

Definition 1. A Besicovitch quasiperiodic function $u(\omega t)$ generated by a function $u(\cdot) \in \mathcal{H}_{\mathcal{A}}$ is called a weak quasiperiodic solution of the natural system on $\mathcal{M}$ if it satisfies the equality

$$\langle D_\omega u(\varphi), D_\omega h(\varphi)\rangle_0 + \langle W'_x(\varphi, u(\varphi)), h(\varphi)\rangle_0 = 0 \tag{1}$$

for any vector field $h(\cdot)$ along $u(\cdot)$.

This definition is natural since the equality (1) holds true for any classical quasiperiodic solution $u(\omega t)$ and continuous vector field $h(\varphi)$ along $u(\cdot)$ with continuous derivative $D_\omega h(\cdot)$. It should be also noted the following fact.

The application of variational approach to the problem of detecting weak quasiperiodic solution consists in finding a function $u_*(\cdot) \in \mathcal{H}_{\mathcal{A}}$ which takes values in appropriately chosen bounded subset $\mathcal{A} \subset \mathcal{M}$ and which is a strong limit in $\mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ of minimizing sequence for the functional (the averaged Lagrangian)

$$J[u] = \int_{\mathbb{T}^k} \left[\frac{1}{2}\|D_\omega u(\varphi)\|^2 + W(\varphi, u(\varphi))\right] d\varphi \tag{2}$$

restricted to $\mathcal{S}_{\mathcal{A}}$. It is naturally to expect that the first variation of J at $u_*(\cdot)$ vanishes, i.e.

$$J'[u_*]\,(h) := \langle D_\omega u_*(\varphi), D_\omega h(\varphi)\rangle_0 + \langle W'_x(\varphi, u_*(\varphi)), h(\varphi)\rangle_0 = 0 \tag{3}$$

for any vector field $h(\cdot)$ along $u_*(\cdot)$. In such a case $u_*(\omega t)$ is a weak quasiperiodic solution.

In order to guarantee the convergence of a minimizing sequence $u_j(\cdot) \in \mathcal{S}_{\mathcal{A}}$ for $J\big|_{\mathcal{S}_{\mathcal{A}}}$ by norm $\|\cdot\|_0$ it is naturally to impose some convexity conditions both on the set $\mathcal{A}$ and on the functional J. Usually, such conditions are formulated by means of geodesics. But in the case where $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ is not a Riemannian manifold of non-positive sectional curvature, we are not able to determine whether the functional of averaged kinetic energy, namely $J_1[u] := \frac{1}{2}\int_{\mathbb{T}^k} \|D_\omega u(\varphi)\|^2 d\varphi$, is convex using geodesics of Levi-Civita connection ∇. if $(\mathcal{M}, \langle\cdot,\cdot\rangle)$. (The case of Riemannian manifold of non-positive sectional curvature was considered in [9, 10].)

In order to overcome the above difficulty we introduce a conformally equivalent inner product of the form $\langle\cdot,\cdot\rangle_V\big|_{T_x\mathcal{M}} := \mathrm{e}^{V(x)}\langle\cdot,\cdot\rangle\big|_{T_x\mathcal{M}}$ with appropriately chosen smooth function $V(\cdot) : \mathcal{M} \mapsto \mathbb{R}$. With this approach we succeed in establishing a required convexity properties of averaged Lagrangian under certain convexity conditions imposed on functions $V(\cdot)$ and $W(\varphi, \cdot)$.

3. Convexity of averaged Lagrangian. It is easily seen that if $V(\cdot) \in \mathrm{C}^\infty(\mathcal{M} \mapsto \mathbb{R})$ is a bounded function on $\mathcal{M}$ then the Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$ equipped with corresponding Levi-Civita connection is complete.

In fact, by definition, the standard distance between any two points $x_1, x_2 \in (\mathcal{M}, \langle\cdot,\cdot\rangle)$ is defined as

$$\rho(x_1, x_2) := \inf\left\{l(c) : c \in \mathcal{C}_{x_1,x_2}\right\},$$

where $\mathcal{C}_{x_1,x_2}$ is the set of all piecewise differentiable paths $c : [0,1] \mapsto \mathcal{M}$ connecting x_1 with x_2, and $l(c)$ is the length of c on $(\mathcal{M}, \langle\cdot,\cdot\rangle)$. If we denote by $l_V(c)$ the length of path c on $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$, then

$$\inf_{x\in\mathcal{M}} \sqrt{\mathrm{e}^{V(x)}}\,l(c) \le l_V(c) \le \sup_{x\in\mathcal{M}} \sqrt{\mathrm{e}^{V(x)}}\,l(c).$$

Hence, the metric $\rho(\cdot,\cdot)$ and the metric $\rho_V(\cdot,\cdot)$ of $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$ are equivalent. Now it remains only to apply the HopfRinow theorem (see, e.g., [13, sect. 5.3]).

In order to distinguish geodesics of metrics ρ and ρ_V we shall call them ρ-geodesic and ρ_V-geodesic respectively.

For $x \in \mathcal{M}$, let $\exp_x(\cdot) : T_x\mathcal{M} \mapsto \mathcal{M}$ denotes the exponential mapping of Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ with Levi-Civita connection ∇ and let $\exp_x^V(\cdot) : T_x\mathcal{M} \mapsto \mathcal{M}$ be the analogous mapping of Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$ with corresponding Levi-Civita connection ∇^V. Note that since both manifolds are complete the domains of both exponential mappings coincide with entire $T_x\mathcal{M}$.

Recall that a set of a Riemannian manifold is called convex if together with any two points x_1, x_2 this set contains a (unique) minimal geodesic segment connecting x_1 with x_2(see, e.g., [12, sect. 11.8] or [13, sect. 5.2]). It is well known that for any point x_0 an open ball of sufficiently small radius centered at point x_0 is convex. A function $f : \mathcal{D}_f \mapsto \mathbb{R}$ with convex domain $\mathcal{D}_f \subset \mathcal{M}$ is convex iff its superposition with any naturally parametrized geodesic containing in $\mathcal{D}_f$ is convex.

Recall also that for the function $V(\cdot)$, the Hesse form $H_V(x)$ at point x (see., e.g., [13]) is defined by the equality

$$\langle H_V(x)\xi, \eta\rangle := \langle \nabla_\xi \nabla V(x), \eta\rangle \quad \forall \xi, \eta \in T_x\mathcal{M}.$$

In addition, let us introduce the following quadratic form

$$\langle G_V(x)\xi, \xi\rangle := \langle H_V(x)\xi, \xi\rangle - \frac{1}{2}\langle \nabla V(x), \xi\rangle^2 \quad \forall \xi \in T_x\mathcal{M},$$

and denote

$$\lambda_V(x) := \min_{\xi\in T_x\mathcal{M}\setminus\{0\}} \langle H_V(x)\xi, \xi\rangle / \|\xi\|^2,$$

$$\mu_V(x) := \min_{\xi\in T_x\mathcal{M}\setminus\{0\}} \langle G_V(x)\xi, \xi\rangle / \|\xi\|^2.$$

We accept the following hypotheses concerning convexity properties of functions $V(\cdot)$ and $W(\cdot)$:

(H1): there exist a bounded function $V(\cdot) \in \mathrm{C}^\infty(\mathcal{M} \mapsto \mathbb{R})$ and a bounded domain $\mathcal{D} \subset \mathcal{M}$ such that

$$\lambda_V(x) + \frac{1}{2}\|\nabla V(x)\|^2 \geq 0, \quad \forall x \in \mathcal{D}; \tag{4}$$

(H2): there exist a noncritical value $v \in V(\mathcal{D})$ and a connected component Ω of open sublevel set $V^{-1}((-\infty, v))$ with the following properties: (a) for any $x, y \in \Omega$ the domain $\mathcal{D}$ contains a unique minimal ρ_V-geodesic segment with endpoints x, y; (b) the second fundamental form of $\partial\Omega$ is positive at each point $x \in \partial\Omega$ (i.e. for any $x \in \partial\Omega$ the restriction of $H_V(x)$ to $T_x\partial\Omega$ is positive definite); (c) the function $V(\cdot)$ satisfies the inequality

$$\mu_V(x) \geq 2K^*(x) \quad \forall x \in \Omega \tag{5}$$

where

$$K^*(x) := \max_{\sigma_x(\xi,\eta)} \frac{\langle R(\eta,\xi)\xi,\eta\rangle}{\|\eta\|^2\|\xi\|^2 - \langle\eta,\xi\rangle^2}$$

is the maximum sectional curvature at point x, R is the Riemann curvature tensor of $(\mathcal{M}, \langle\cdot,\cdot\rangle)$, $\sigma_x(\xi,\eta)$ is a plane defined by vectors $\xi, \eta \in T_x\mathcal{M}$, and $K(\sigma_x(\xi,\eta))$ is a sectional curvature in direction $\sigma_x(\xi,\eta)$ [13];

(H3): the function $W(\cdot,\cdot)$ satisfies the following inequalities

$$\lambda_W(\varphi,x) + \frac{1}{2}\langle\nabla W(\varphi,x), \nabla V(x)\rangle > 0 \quad \forall(\varphi,x) \in \mathbb{T}^k \times \bar{\Omega} \quad (\bar{\Omega} := \Omega \cup \partial\Omega),$$
$$\langle\nabla W(\varphi,x), \nabla V(x)\rangle > 0 \quad \forall(\varphi,x) \in \mathbb{T}^k \times \partial\Omega$$

where $\lambda_W(\varphi,x)$ is minimal eigenvalue of Hesse form $H_W(\varphi,x)$ for the function $W(\varphi,\cdot) : \mathcal{M} \mapsto \mathbb{R}$.

Theorem 1. *Let the Hypotheses (H1)–(H3) hold true. Then there exist positive constants C, C_1 and c such that for any $u_0(\cdot), u_1(\cdot) \in \mathrm{C}^\infty(\mathbb{T}^k \mapsto \Omega)$ one can choose a vector field $h(\cdot) \in \mathrm{C}^\infty(\mathbb{T}^k \mapsto T\mathcal{M})$ along $u_0(\cdot)$ (this implies that $h(\varphi) \in T_{u_0(\varphi)}\mathcal{M}$ for all $\varphi \in \mathbb{T}^k$) in such a way that the following inequalities hold true*

$$c\rho(u_0(\varphi), u_1(\varphi)) \leq \|h(\varphi)\| \leq C\rho(u_0(\varphi), u_1(\varphi)) \quad \forall\varphi \in \mathbb{T}^k,$$
$$\|D_\omega h(\varphi)\| \leq C_1\left[\|D_\omega u_0(\varphi)\| + \|D_\omega u_1(\varphi)\|\right] \quad \forall\varphi \in \mathbb{T}^k,$$
$$J[u_1] - J[u_0] - J'[u_0](h) \geq \frac{\varkappa c^2}{2}\int\limits_{\mathbb{T}^k} \rho^2(u_0, u_1)\mathrm{d}\varphi$$

where $\varkappa := \min\left\{\lambda_W(\varphi,x) + \frac{1}{2}\langle\nabla W(\varphi,x), \nabla V(x)\rangle : (\varphi,x) \in \mathbb{T}^k \times \bar{\Omega}\right\}.$

The proof of this theorem needs several auxiliary statements and will be given below at the end of present Section.

Proposition 1. *The Euler-Lagrange equation for ρ_V-geodesic on Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ has the form*

$$\nabla_{\dot{x}}\dot{x} = -\langle\nabla V(x), \dot{x}\rangle\,\dot{x} + \frac{\|\dot{x}\|^2}{2}\nabla V(x), \tag{6}$$

Proof. A ρ_V-geodesic segment with endpoints $x_0, x_1 \in \mathcal{M}$ is an extremal of functional $\Phi[x(\cdot)] = \int_0^1 \mathrm{e}^{V\circ x(t)}\,\|\dot{x}(t)\|^2\,\mathrm{d}t$ defined on the space $\mathcal{C}^2_{x_0x_1}$ of twice continuous differentiable curves $x = x(t)$, $t \in [0,1]$, such that $x(0) = x_0$, $x(1) = x_1$. We are going to derive the Euler-Lagrange equation using the connection ∇. Consider a variation of $x(\cdot)$ defined by a smooth mapping $y(\cdot,\cdot) : [0,1]\times(-\varepsilon,\varepsilon) \mapsto \mathcal{M}$ such that $y(\cdot,\lambda) \in \mathcal{C}^\infty_{x_0x_1}$ for any fixed $\lambda \in (-\varepsilon,\varepsilon)$ and $y(t,0) \equiv x(t)$. Put

$$\dot{y}(t,\lambda) := \frac{\partial}{\partial t}y(t,\lambda), \quad y'(t,\lambda) := \frac{\partial}{\partial\lambda}y(t,\lambda).$$

Obviously, $\dot{y}(t,0) = \dot{x}(t)$, $y(i,\lambda) \equiv x_i$, and $y'(i,\lambda) = 0$, $i = 0,1$. Then since $\nabla_{y'}\dot{y} = \nabla_{\dot{y}}y'$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\int_0^1 \mathrm{e}^{V\circ y}\,\|\dot{y}\|^2\,\mathrm{d}s =$$

$$= \int_0^1 \left[\mathrm{e}^{V\circ y}\left\langle\nabla V\circ y, y'\right\rangle\|\dot{y}\|^2 + 2\mathrm{e}^{V\circ y}\left\langle\nabla_{y'}\dot{y},\dot{y}\right\rangle\right]_{\lambda=0}\mathrm{d}t =$$

$$= \int_0^1 \left[\mathrm{e}^{V\circ y}\left\langle\nabla V\circ y, y'\right\rangle\|\dot{y}\|^2 + 2\mathrm{e}^{V\circ y}\left\langle\nabla_{\dot{y}}y',\dot{y}\right\rangle\right]_{\lambda=0}\mathrm{d}t.$$

Taking into account that

$$\frac{\partial}{\partial t}\mathrm{e}^{V\circ y}\left\langle y',\dot{y}\right\rangle = \mathrm{e}^{V\circ y}\left\langle\nabla V\circ y,\dot{y}\right\rangle\left\langle y',\dot{y}\right\rangle + \mathrm{e}^{V\circ y}\left\langle\nabla_{\dot{y}}y',\dot{y}\right\rangle + \mathrm{e}^{V\circ y}\left\langle y',\nabla_{\dot{y}}\dot{y}\right\rangle$$

and $\mathrm{e}^{V\circ y}\left\langle y',\dot{y}\right\rangle\big|_{t=0,1} = 0$, we get

$$\int_0^1 \mathrm{e}^{V\circ y}\left\langle\nabla_{\dot{y}}y',\dot{y}\right\rangle\mathrm{d}t = -\int_0^1 \mathrm{e}^{V\circ y}\left[\left\langle\nabla V\circ y,\dot{y}\right\rangle\left\langle y',\dot{y}\right\rangle + \left\langle y',\nabla_{\dot{y}}\dot{y}\right\rangle\right]\mathrm{d}t.$$

From this it follows that the first variation on functional Φ is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\Phi[y(\cdot,\lambda)] = \Phi'[x(\cdot)]\left(y'(\cdot,0)\right) =$$

$$= \int_0^1 \left[\mathrm{e}^V\left(\left\langle\nabla V, y'\right\rangle\|\dot{x}\|^2 - 2\left\langle\nabla V,\dot{x}\right\rangle\left\langle\dot{x},y'\right\rangle - 2\left\langle\nabla_{\dot{x}}\dot{x},y'\right\rangle\right)\right]\Big|_{x=x(t),\lambda=0}\mathrm{d}t,$$

and the Euler-Lagrange equation is exactly (6). ∎

Proposition 2. *Let the Hypothesis (H1) holds true. If a ρ_V-geodesic segment connecting points x_0, x_1 of the set Ω belongs to $\mathcal{D}$, then this segment belongs to Ω.*

Proof. Let $x(\cdot) \in \mathcal{C}^2_{x_0x_1}$ satisfies (6) and let $x(t) \in \mathcal{D}$ for all $t \in [0,1]$. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{e}^V\Big|_{x=x(t)} =$$

$$= \left[\mathrm{e}^V \left(\langle \nabla_{\dot{x}} \nabla V, \dot{x}\rangle + \left\langle \nabla V, -\langle \nabla V, \dot{x}\rangle \dot{x} + \|\dot{x}\|^2 \nabla V/2\right\rangle + \langle \nabla V, \dot{x}\rangle^2\right)\right] \Big|_{x=x(t)} =$$

$$= \left[\mathrm{e}^V \left(\langle \nabla_{\dot{x}} \nabla V, \dot{x}\rangle + \|\dot{x}\|^2 \|\nabla V\|^2 /2\right)\right] \Big|_{x=x(t)} \geq$$

$$\geq \left[\mathrm{e}^V \|\dot{x}\|^2 \left(\lambda_V + \|\nabla V\|^2 /2\right)\right] \Big|_{x=x(t)} \geq 0.$$

Hence, $\mathrm{e}^{V \circ x(\cdot)}$ is convex and this implies $V \circ x(t) < v$ for all $t \in [0,1]$. $\square$

Proposition 3. *Under the Hypotheses (H1)-(H2), the minimal ρ_V-geodesic segment connecting any two points $x, y \in \Omega$ does not contain conjugate points.*

Proof. It is known (see. [13, sect. 3.6]) that the sectional curvature in direction $\sigma_x(\xi_1, \xi_2)$ on Riemannian manifold $\left(\mathcal{M}, \mathrm{e}^V \langle \cdot, \cdot \rangle\right)$ is represented in the form

$$K_V\left(\sigma_x(\xi_1,\xi_2)\right) = \mathrm{e}^{-V} K(\sigma_x(\xi_1,\xi_2)) -$$

$$-\frac{\mathrm{e}^{-V}}{2} \sum_{i=1}^{2} \left[\langle H_V(x)\xi_i, \xi_i\rangle - \frac{1}{2}\langle \nabla V(x), \xi_i\rangle^2\right] - \frac{\mathrm{e}^{-V}}{4} \|\nabla V(x)\|^2$$

where ξ_1, ξ_2 is an orthonormal basis of the plane $\sigma_x(\xi_1, \xi_2)$, and the inequality (5) yields that this curvature is non-positive for any $x \in \bar{\Omega}$. By the Morse–Schoenberg theorem any ρ_V-geodesic segment containing in $\bar{\Omega}$ does not contain conjugate points. $\square$

Proposition 4. *Under the Hypotheses (H1)-(H3) there exists a smooth mapping $\zeta(\cdot,\cdot) : \Omega \times \Omega \mapsto T\mathcal{M}$ such that $\zeta(x,y) \in T_x\mathcal{M}$ and*

$$\exp^V_x(\zeta(x,y)) = y, \quad \mathrm{e}^{V(x)/2} \|\zeta(x,y)\| = \rho_V(x,y), \tag{7}$$

$$\exp^V_x(t\zeta(x,y)) \in \Omega \quad \forall t \in [0,1]. \tag{8}$$

Proof. It is known that if for some $\xi \in T_x\mathcal{M}$ a geodesic segment $\exp^V_x(t\xi)$, $t \in [0,1]$, does not contain conjugate points then the mapping $\exp^V_x(\cdot)$ is local diffeomorphism at any point $t\xi$, $t \in [0,1]$. Under the Hypothesis (H2) for any $x, y \in \Omega$ there exists a unique $\zeta(x,y)$ which satisfies conditions (8). It follows from the implicit function theorem that the mapping $\zeta(\cdot,\cdot) : \Omega \times \Omega \mapsto T\mathcal{M}$ is smooth. $\square$

If we define the mapping

$$\gamma_V(\cdot,\cdot,\cdot) : [0,1] \times \Omega \times \Omega \mapsto \Omega, \quad \gamma_V(t,x,y) := \exp^V_x(t\zeta(x,y)),$$

then for any $x, y \in \mathcal{D}$ the mapping $\gamma_V(\cdot, x, y) : [0,1] \mapsto \mathcal{D}$ satisfies the equation (6) together with boundary conditions $\gamma_V(0,x,y) = x$, $\gamma_V(1,x,y) = y$. The following scalar differential equation

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = \exp\left(V \circ \gamma_V(\tau, x, y)\right) \int\limits_0^1 \exp\left(-V \circ \gamma_V(t,x,y)\right) \mathrm{d}t.$$

has a unique strictly monotonically increasing solution

$$\tau(\cdot, x, y) : [0,1] \mapsto [0,1], \quad \tau(0,x,y) = 0, \quad \tau(1,x,y) = 1. \tag{9}$$

By means of reparametrisation $t = \tau(s,x,y)$ we define a smooth mapping

$$\chi(\cdot,\cdot,\cdot) : [0,1] \times \Omega \times \Omega \mapsto \Omega, \quad \chi(s,x,y) := \gamma_V(\tau(s,x,y), x, y)$$

which plays an important role in subsequent reasoning. In [7] $\chi(\cdot,\cdot,\cdot)$ is called the connecting mapping.

Proposition 5. *For any $x, y \in \Omega$ the mapping $\chi(\cdot, x, y) : [0,1] \mapsto \Omega$ satisfies the equation*

$$\nabla_{x'} x' = \frac{\|x'\|^2}{2} \nabla V(x), \tag{10}$$

where $x' = \frac{\mathrm{d}x}{\mathrm{d}s}$ and the boundary conditions $\chi(0,x,y) = x$, $\chi(1,x,y) = y$.

Proof. The boundary conditions follow from definition of γ_V and (9). Let us show that (10) is obtained from (6) after the change of independent variable $t = \tau(s)$. In fact, let $\chi(s) = x \circ \tau(s)$. Then (6) takes the form

$$\frac{1}{\tau'} \nabla_{\chi'} \left(\frac{1}{\tau'} \chi'\right) = -\frac{1}{(\tau')^2} \left\langle \nabla V \circ \chi, \chi' \right\rangle \chi' + \frac{\|\chi'\|^2}{2(\tau')^2} \nabla V \circ \chi,$$

or

$$-\frac{\tau''}{\tau'} \chi' + \nabla_{\chi'} \chi' = -\left[\frac{\mathrm{d}}{\mathrm{d}s} V \circ \chi\right] \chi' + \frac{\|\chi'\|^2}{2} \nabla V \circ \chi.$$

From this it follows (10) since $\tau''/\tau' = (V \circ \chi)'$. □

Proposition 6. *Let $u_i(\cdot) \in \mathcal{S}_\Omega$, $i = 0,1$. Then under the hypotheses (H1)-(H2) the following inequality is valid*

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \left\| D_\omega \chi\left(s, u_0(\varphi), u_1(\varphi)\right) \right\|^2 \geq 0 \quad \forall s \in [0,1], \ \forall \varphi \in \mathbb{T}^k.$$

Proof. For any fixed $\varphi \in \mathbb{T}^k$ put

$$\eta(s,t) := \frac{\partial}{\partial t} \chi\left(s, u_0(\varphi + \omega t), u_1(\varphi + \omega t)\right) \equiv D_\omega \chi\left(s, u_0(\varphi + \omega t), u_1(\varphi + \omega t)\right),$$

$$\xi(s,t) := \frac{\partial}{\partial s} \chi\left(s, u_0(\varphi + \omega t), u_1(\varphi + \omega t)\right).$$

Then in view of the well known relations (see. e.g., [13],DNF84)

$$\nabla_\eta \xi = \nabla_\xi \eta, \quad \nabla_\eta \nabla_\xi \xi - \nabla_\xi \nabla_\eta \xi = R(\eta, \xi)\xi$$

and (10), we have

$$\nabla_\xi^2 \eta = \nabla_\eta \nabla_\xi \xi - R(\eta,\xi)\xi =$$
$$= \langle \nabla_\eta \xi, \xi \rangle \nabla V \circ \chi + \frac{\|\xi\|^2}{2} \nabla_\eta \nabla V \circ \chi - R(\eta,\xi)\xi$$

and hence,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \|\eta\|^2 = 2\left[\left\langle \nabla_\xi^2 \eta, \eta \right\rangle + \|\nabla_\xi \eta\|^2\right] =$$
$$= 2\|\nabla_\xi \eta\|^2 + 2\langle \nabla_\xi \eta, \xi\rangle \langle \nabla V \circ \chi, \eta \rangle +$$
$$+ \|\xi\|^2 \langle \nabla_\eta \nabla V \circ \chi, \eta \rangle - 2\langle R(\eta,\xi)\xi, \eta\rangle \geq$$
$$\geq 2\|\nabla_\xi \eta\|^2 - 2\|\nabla_\xi \eta\| \|\xi\| |\langle \nabla V \circ \chi, \eta\rangle| +$$
$$+ \|\xi\|^2 \langle \nabla_\eta \nabla V \circ \chi, \eta\rangle - 2K^* \circ \chi \|\xi\|^2 \|\eta\|^2 .$$

Once the Hypothesis (H2) holds true, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \|\eta\|^2 \geq$$
$$\geq 2\|\xi\|^2 \|\eta\|^2 \left[r^2 - |\langle \nabla V \circ \chi, \mathbf{e}\rangle| r + \frac{1}{2}\langle \nabla_{\mathbf{e}} \nabla V \circ \chi, \mathbf{e}\rangle - K^* \circ \chi\right] \geq 0$$

where $r := \frac{\|\nabla_\xi \eta\|}{\|\xi\|\|\eta\|}$. □

Now we are in position to prove the Theorem 1. Let $u_i(\cdot) \in \mathcal{S}_\Omega$, $i = 0, 1$. By means of connecting mapping we get the following representation

$$J[\chi(s,u_0,u_1)] = J[u_0] + sJ'[u_0]\left(\chi_s'(0,u_0,u_1)\right) + \frac{s^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=\theta} J\left[\chi\left(s,u_0,u_1\right)\right] \tag{11}$$

with some $\theta \in (0,1)$. To estimate from below the term with second derivative we make use of Proposition 6 which together with the Hypothesis (H3) implies

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[\frac{1}{2}\|D_\omega \chi\left(s, u_0(\varphi), u_1(\varphi)\right)\|^2 + W\left(\varphi, \chi(s,u_0,u_1)\right)\right] \geq$$
$$\geq \frac{\mathrm{d}}{\mathrm{d}s}\left\langle \nabla W(\varphi,\chi), \chi_s'\right\rangle = \left\langle \nabla_{\chi_s'} \nabla W(\varphi,\chi), \chi_s'\right\rangle + \left\langle \nabla W(\varphi,\chi), \nabla_{\chi_s'}\chi_s'\right\rangle =$$
$$= \left\langle \nabla_{\chi_s'} \nabla W(\varphi,\chi), \chi_s'\right\rangle + \frac{\|\chi_s'\|^2}{2}\langle \nabla W(\varphi,\chi), \nabla V(\chi)\rangle \geq \varkappa \left\|\chi_s'\right\|^2 .$$

By the definition of χ we have

$$\chi_s'\left(s,u_0,u_1\right) = \tau'(s)\dot\gamma_V\left(\tau(s),u_0,u_1\right) =$$
$$= \exp\left(V \circ \gamma_V(\tau(s),u_0,u_1)\right)\int_0^1 \exp\left(-V \circ \gamma_V(t,u_0,u_1)\right)\mathrm{d}t\,\dot\gamma_V\left(\tau(s),u_0,u_1\right).$$

Since $\gamma_V(t,x,y)$ is ρ_V-geodesic, then $\exp\left(V\circ\gamma_V\right)\|\dot\gamma_V\|^2$ does not depend on t and

$$\mathrm{e}^{V(x)/2}\left\|\dot\gamma_V(0,x,y)\right\| = \mathrm{e}^{V(x)/2}\left\|\zeta(x,y)\right\| = \rho_V(x,y).$$

Hence

$$\left\|\chi'_s\left(s,u_0,u_1\right)\right\|^2 = \left[\int\limits_0^1 \exp\left(-V\circ\gamma_V(t,u_0,u_1)\right)\mathrm{d}t\right]^2 \times$$
$$\times \exp\left(V\circ\gamma_V(\tau(s),u_0,u_1)\right)\rho_V^2(u_0,u_1),$$

and (8) implies that there exist positive constants C, c dependent only on $V(\cdot)$ and Ω such that

$$c\rho(u_0,u_1) \le \left\|\chi'_s\left(s,u_0,u_1\right)\right\| \le C\rho(u_0,u_1). \tag{12}$$

Define $h(\varphi) := \chi'_s\left(0,u_0(\varphi),u_2(\varphi)\right)$. Then (11) with $s=1$ yields

$$J[u_1]-J[u_0]-J'[u_0]\left(\chi'_s(0,u_0,u_1)\right) \ge \frac{\varkappa c^2}{2}\int\limits_{\mathbb{T}^k}\rho^2(u_0,u_1)\mathrm{d}\varphi.$$

Finally, since the set Ω is bounded and the mapping χ is smooth, there exists positive constant C_1 such that

$$\left\|D_\omega h(\varphi)\right\| \le C_1\left[\left\|D_\omega u_0(\varphi)\right\| + \left\|D_\omega u_1(\varphi)\right\|\right] \quad \forall\varphi\in\mathbb{T}^k.$$

The proof of Theorem 1 is complete.

4. Main existence theorem. Now we proceed to the main result of this paper.

Theorem 2. *Let the Hypotheses (H1)–(H3) hold true. Then the natural system on Riemannian manifold $(\mathcal{M},\langle\cdot,\cdot\rangle)$ with Lagrangian density $L = \frac{1}{2}\langle\dot x,\dot x\rangle + W(\omega t,x)$ has a weak quasiperiodic solution.*

Proof. The proof will consist of three steps.

1. Construction of a projection mapping and its smooth approximation. Put $\Omega+\delta = \left(\bigcup_{x\epsilon\Omega}B(x;\delta)\right)$ where $B(x;\delta)$ stands for an open ball of radius δ centered at $x\in\mathcal{M}$ on Riemannian manifold $(\mathcal{M},\langle\cdot,\cdot\rangle)$. Since by Hypothesis (H2) v is a noncritical value, then $\partial\Omega = V^{-1}(v)$ is a regular hypersurface with unit normal field $\boldsymbol{\nu} := \frac{\nabla V}{\|\nabla V\|}$. As is well known (see, e.g., [12]), for sufficiently small $\delta>0$, one can correctly define the projection mapping $P_\Omega : \Omega+\delta \to \bar\Omega$ such that $P_\Omega x\in\bar\Omega$ is the nearest point to $x\in\Omega+\delta$. If $x = X(q)$, $q\in\mathcal{Q}\subset\mathbb{R}^{m-1}$, is a smooth local parametric representation of $\partial\Omega$ in a neighborhood of a point $x_0\in\partial\Omega$, then for sufficiently small $\delta_0>0$ the mapping

$$\mathcal{Q}\times(-\delta_0,\delta_0)\ni(q,z)\mapsto\exp_{X(q)}\left(z\boldsymbol{\nu}\circ X(q)\right)$$

introduces local coordinates with the following properties: local equation of $\partial\Omega$ is $z=0$; each naturally parametrized ρ-geodesic $\gamma(s) =$

$\exp_{X(q)}(s\boldsymbol{\nu}\circ X(q))$ is orthogonal to each hypersurface $z=\mathrm{const}$; the Riemannian metric takes the form $\sum_{i,j=1}^{m-1} b_{ij}(q,z)\mathrm{d}q_i\mathrm{d}q_j+\mathrm{d}z^2$, where $B(q,z)=\{b_{ij}(q,z)\}_{i,j=1}^{m-1}$ is positive definite symmetric matrix; the function $V(\cdot)$ is represented in the form $V(q,z)=v+a(q)z+b(q,z)z^2$; the mapping P_Ω has the form

$$P_\Omega(q,z):=\begin{cases}(q,0) & \text{if} \quad z\in(0,\delta_0),\\ (q,z) & \text{if} \quad z\in(-\delta_0,0].\end{cases}$$

The projection mapping is continuous on $\Omega+\delta$ and continuously differentiable on $(\Omega+\delta)\backslash\partial\Omega$. Moreover, it turns out that for sufficiently small $\delta>0$ the derivative $P_{\Omega*}$ is contractive on $(\Omega+\delta)\backslash\partial\Omega$, i.e.

$$\|P_{\Omega*}\xi\|\le\|\xi\| \quad \forall\xi\in T_x\mathcal{M},\ x\in(\Omega+\delta)\backslash\partial\Omega. \tag{13}$$

It is sufficiently to prove this inequality for any $x\in(\Omega+\delta)\backslash\partial\Omega$. Let $q=q(s)$, $z=z(s)$ be natural equations of ρ-geodesic which starts at a point $x_0=(q_0,0)\in\partial\Omega$ in direction of vector $\eta=(\dot q_0,0)\in T_{x_0}\partial\Omega$. The hypothesis (H2) implies that

$$\langle\nabla_\eta\nabla V(x_0),\eta\rangle=\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0}V(q(s),z(s))>0 \quad\Leftrightarrow\quad a(q_0)\ddot z(0)>0.$$

Since $a(q_0)>0$ ($\boldsymbol{\nu}$ is external normal to $\partial\Omega$) and z-component of geodesic equations is

$$\ddot z=\frac{1}{2}\frac{\partial}{\partial z}\sum_{i,j=1}^{m-1}b_{ij}(q,z)\dot q_i^2\dot q_j^2,$$

then the matrix $B'_z(q_0,0)$ is positive definite. From this it follows that $B(q,z_1)>B(q,z_2)$ for all q from a neighborhood of q_0 and all $z_1,z_2\in(-\delta,\delta)$, $z_1>z_2$ if $\delta\in(0,\delta_0)$ is sufficiently small. Let $\xi=(\dot q,\dot z)$ be a tangent vector at point (q,z) where $z\in(0,\delta)$. Then

$$\|\xi\|^2=\sum_{i,j=1}^{m-1}b_{ij}(q,z)\dot q_i\dot q_j+\dot z^2\ge$$
$$ge\sum_{i,j=1}^{m-1}b_{ij}(q,z)\dot q_i\dot q_j\ge\sum_{i,j=1}^{m-1}b_{ij}(q,0)\dot q_i\dot q_j=\|(\dot q,0)\|^2=\|P_{\Omega*}\xi\|^2.$$

Let us introduce a smooth approximation of projection mapping in a following way. For $\varepsilon\in(0,\delta)$ define

$$\varpi_\varepsilon(z):=\begin{cases}\exp\left(1/z-1/(z+\varepsilon)\right), & z\in(-\varepsilon,0),\\ 0, & z\in\mathbb{R}\setminus(-\varepsilon,0),\end{cases}$$

$$Z_\varepsilon(z):=\int\limits_{-\varepsilon}^{z}\frac{\int_s^0\varpi_\varepsilon(t)\mathrm{d}t}{\int_{-\varepsilon}^0\varpi_\varepsilon(t)\mathrm{d}t}\mathrm{d}s-\varepsilon, \quad z\in(-\delta_0,\delta_0)$$

Obviously that the function $Z_\varepsilon(\cdot)$ is smooth, its derivative, $Z'_\varepsilon(z)$, equals 1 for $z \in (-\delta_0, -\varepsilon]$, monotonically decreases from 1 to 0 on $[-\varepsilon, 0]$, and equals 0 for $z \geq 0$. From this it follows that $Z_\varepsilon(z)$ equals z for $z \in (-\delta_0, -\varepsilon]$ monotonically increases from $-\varepsilon$ to $Z_\varepsilon(0) \in (-\varepsilon, 0)$ on $[-\varepsilon, 0]$, and equals $Z_\varepsilon(0)$ for $z \in [0, \delta_0)$. Now locally define

$$P_{\varepsilon,\Omega}(q, z) := \begin{cases} (q, Z_\varepsilon(0)) & \text{if} \quad z \in (0, \delta_0), \\ (q, Z_\varepsilon(z)) & \text{if} \quad z \in (-\delta_0, 0] \end{cases}$$

and for each point $x \in \Omega$ such that $B(x; \delta) \subset \Omega$ put $P_{\varepsilon,\Omega}(x) = x$. Since $Z_\varepsilon(0) < 0$, then

$$P_{\varepsilon,\Omega}(\Omega + \delta) \subset \Omega$$

and since $|Z'_\varepsilon(z)| \leq 1$, then for any $z \in (-\delta, \delta)$, and for any tangent vector $\xi = (\dot{q}, \dot{z})$ at point (q, z) we have

$$\|\xi\|^2 = \sum_{i,j=1}^{m-1} b_{ij}(q, z)\dot{q}_i\dot{q}_j + \dot{z}^2 \geq \sum_{i,j=1}^{m-1} b_{ij}(q, Z_\varepsilon(z))\dot{q}_i\dot{q}_j + \left(Z'_\varepsilon(z)\dot{z}\right)^2 =$$

$$= \left\|(\dot{q}, Z'_\varepsilon(z)\dot{z})\right\|^2 = \|P_{\varepsilon,\Omega*}\xi\|.$$

From this it follows that

$$\|P_{\varepsilon,\Omega*}\xi\| \leq \|\xi\| \quad \forall x \in \Omega + \delta, \ \forall \xi \in T_x\mathcal{M}. \tag{14}$$

Besides, the Hypothesis (H3) implies

$$W(\varphi, P_{\varepsilon,\Omega}x) \leq W(\varphi, x) \quad \forall \varphi \in \mathbb{T}^m, \forall x \in \Omega + \delta \tag{15}$$

for sufficiently small δ and $\varepsilon \in (0, \delta)$.

2. Minimization of functional J on $\mathcal{S}_{\Omega+\delta}$. Obviously that the functional J restricted to $S_{\Omega+\delta}$ is bounded from below. Let us show that

$$J_* := \inf J[\mathcal{S}_{\Omega+\delta}] = \inf J[\mathcal{S}_\Omega]. \tag{16}$$

In fact, if $v_j(\cdot) \in \mathcal{S}_{\Omega+\delta}$ is such a sequence that $J[v_j]$ monotonically decreases to J_*, then (14) and (15) implies

$$J_* \leq J[P_{\varepsilon/j,\Omega}v_j] \leq J[v_j].$$

Hence, the sequence $u_j(\cdot) := P_{\varepsilon/j,\Omega}v_j(\cdot)$ is minimizing both for $J\big|_{S_\Omega}$ and for $J\big|_{\mathcal{S}_{\Omega+\delta}}$.

3. Convergence of minimizing sequence to a weak solution. Let $u_j(\cdot) \in \mathcal{S}_\Omega$ be a minimizing sequence for $J\big|_{\mathcal{S}_\Omega}$. Without loss of generality, we may consider that

$$\|D_\omega u_j\|_0^2 \leq M := \frac{2}{(2\pi)^k} \sup_{x\in\Omega} \int_{\mathbb{T}^k} W(\varphi, x)\mathrm{d}\varphi - \frac{2}{(2\pi)^k} \int_{\mathbb{T}^k} \inf_{x\in\Omega} W(\varphi, x)\mathrm{d}\varphi. \tag{17}$$

Let $h_j(\cdot) \in \mathrm{C}^\infty\left(\mathbb{T}^k \mapsto T\mathcal{M}\right)$ be a sequence of smooth mappings such that $h_j(\varphi) \in T_{u_j(\varphi)}\mathcal{M}$ for any $\varphi \in \mathbb{T}^k$ and besides there exist positive constants $K,\ K_1$ such that

$$\|h_j\|_1 \le K_1, \quad \|h_j(\varphi)\| \le K \quad \forall \varphi \in \mathbb{T}^k, \quad \forall j = 1, 2, \ldots \tag{18}$$

Let us show that

$$\lim_{j\to\infty} J'[u_j](h_j) = 0. \tag{19}$$

On one hand, $J[u_j]$ decreases to $J_* := \inf J[\mathcal{S}_\Omega]$. On the other hand, for sufficiently small $s_0 \le 1$ and for any $j \in \mathbb{N}$ there exists a number $\theta_j \in [-s_0, s_0]$ such that

$$J[\exp_{u_j}(sh_j)] = J[u_j] + sJ'[u_j](h_j) + \frac{s^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=\theta_j} J[\exp_{u_j}(sh_j)]$$
$$\forall s \in [-s_0, s_0], \quad \forall j \in \mathbb{N},$$

and, besides, there exists a constant $K_2 > 0$ such that

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}s^2} J[\exp_{u_j}(sh_j)]\right| \le K_2 \quad \forall s \in [-s_0, s_0], \quad \forall j \in \mathbb{N}.$$

If now we suppose that $\limsup_{j\to\infty} |J'[u_j](h_j)| > 0$ then one can choose j and $s_j \in [-s_0, s_0]$ in such a way that

$$\exp_{u_j}(s_j h_j) \in \mathcal{S}_{\Omega+\delta}, \quad J[\exp_{u_j}(s_j h_j)] < J_*.$$

Thus, in view of (16), we arrive at contradiction with definition of J_*.

Now by Theorem 1 for any pair $u_{i+j}(\cdot),\ u_j(\cdot)$ there exists a vector field $h_{ij}(\cdot)$ along $u_j(\cdot)$ such that

$$J[u_{i+j}] - J[u_j] - J'[u_j](h_{ij}) \ge \frac{\varkappa c^2}{2} \int\limits_{\mathbb{T}^k} \rho^2(u_j, u_{i+j})\mathrm{d}\varphi \ge$$
$$\ge \frac{(2\pi)^k \varkappa c^2}{2} \|u_{i+j} - u_j\|_0^2 .$$

Since (19) implies $J'[u_j](h_{ij}) \to 0$ as $j \to \infty$, then the sequence $u_j(\cdot)$ is fundamental in $\mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ and in view of (17) converges to a function $u_*(\cdot)$ strongly in $\mathrm{H}(\mathbb{T}^k \mapsto \mathbb{E}^n)$ and weakly in $\mathrm{H}^1_\omega(\mathbb{T}^k \mapsto \mathbb{E}^n)$. Without loss of generality we may consider that $u_*(\cdot)$ is defined by a minimizing sequence which converges a.e.

Now it remains only to prove that $u_*(\cdot)$ is a weak solution, i.e. that there holds (3). Let $h(\cdot)$ be a vector field along $u_*(\cdot)$. By definition, there exists a sequence of smooth mappings $h_j(\varphi) \in T_{u_j(\varphi)}\mathcal{M}$ which satisfies (18) and (19). Then, in view of (17), we get

$$\lim_{j\to\infty} \left|\langle D_\omega u_*, D_\omega h\rangle_0 - \langle D_\omega u_j, D_\omega h_j\rangle_0\right| \le$$
$$\le \lim_{j\to\infty} \left|\langle D_\omega\left(u_* - u_j\right), D_\omega h\rangle_0\right| + \sqrt{M} \lim_{j\to\infty} \|D_\omega\left(h - h_j\right)\|_0 = 0,$$

and by the Lebesgue theorem

$$\lim_{j\to\infty}\int\limits_{\mathbb{T}^k}\left[W\left(\varphi,u_j(\varphi)\right)-W\left(\varphi,u_*(\varphi)\right)\right]\mathrm{d}\varphi=0.$$

Hence,

$$J'[u_*](h)=\lim_{j\to\infty}J'[u_j](h_j)=0.$$

□

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